

Fractional smoothness of functionals of diffusion processes under a change of measure

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Abstract

Let $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the solution of the parabolic backward equation $\partial_t v + (1/2) \sum_{i,l} [\sigma \sigma^\perp]_{il} \partial_{x_i} \partial_{x_l} v + \sum_i b_i \partial_{x_i} v + kv = 0$ with terminal condition g , where the coefficients are time- and state-dependent, and satisfy certain regularity assumptions. Let $X = (X_t)_{t \in [0, T]}$ be the

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associated \mathbb{R}^d -valued diffusion process on some appropriate $(\Omega, \mathcal{F}, \mathbb{Q})$. For $p \in [2, \infty)$ and a measure $d\mathbb{P} = \lambda_T d\mathbb{Q}$, where λ_T satisfies the Muckenhoupt condition A_α for $\alpha \in (1, p)$, we relate the behavior of $\|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)\|_{L_p(\mathbb{P})}$, $\|\nabla v(t, X_t)\|_{L_p(\mathbb{P})}$ and $\|D^2 v(t, X_t)\|_{L_p(\mathbb{P})}$ to each other, where $D^2 v := (\partial_{x_i} \partial_{x_l} v)_{i,l}$ is the Hessian matrix.

1 Introduction

For a fixed time-horizon $T > 0$ let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{Q})$ be a filtered probability space where $(\Omega, \mathcal{F}, \mathbb{Q})$ is complete, $\mathcal{F} = \mathcal{F}_T$, the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is right-continuous, \mathcal{F}_0 is generated by the null sets of \mathcal{F} and where all local martingales are continuous (see Section 2). Assume for some $d \geq 1$ that the process $B = (B_t)_{t \in [0, T]}$ is a d -dimensional $(\mathcal{F}_t)_{t \in [0, T]}$ -standard Brownian motion starting in zero. We consider an \mathbb{R}^d -valued diffusion process $X = (X_t)_{t \in [0, T]}$, solution to the stochastic differential equation

$$X_t = x_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds$$

for some smooth bounded coefficients b and σ , and we focus on the rate of convergence of

$$R_p^X(t) := \|g(X_T) - \mathbb{E}(g(X_T)|\mathcal{F}_t)\|_p$$

for $p \in [2, \infty)$ as $t \rightarrow T$, where g satisfies a suitable growth condition ensuring $g(X_T) \in L_p$. The behavior of $R_p^X(t)$ as $t \rightarrow T$ is a measure of the fractional smoothness of g , see [4] for an overview. Actually it is now well-known [3, 6, 10, 5] that there is a precise correspondence between the irregularity of the terminal function g and the time-singularity of the L_p -norms of $\nabla v(t, X_t)$ as $t \uparrow T$ where

$$v(t, x) = \mathbb{E}(g(X_T)|X_t = x).$$

The aim of this paper is to extend these quantitative equivalence results to situations where the L_p -norms are computed under different measures. The theory of probabilistic Muckenhoupt weights, developed as a counterpart to the deterministic ones from [14] and other papers, gives a natural way to extend various martingale inequalities to equivalent measures, see exemplary [12, 1, 13] and the references therein. A typical situation is a change of measure initiated by a Girsanov transformation, i.e. a change of the drift of X . Applying the results of this paper in this particular case, gives -without

going into full details- the following: if the process Y differs from X by another bounded drift and if $\theta \in (0, 1)$, then we have

$$\sup_{t \in [0, T)} (T - t)^{-\theta/2} R_p^Y(t) < \infty \iff \sup_{t \in [0, T)} (T - t)^{(1-\theta)/2} \|\nabla v(t, Y_t)\|_p < +\infty \quad (1)$$

which follows from Theorem 1 below for $q = \infty$ as explained in Remark 2(7). The parameter θ is the degree of *fractional smoothness*.

Regarding the references in the literature related to (1), a 1-dimensional diffusion case with $X = Y$ is considered in [3], the extension to multidimensional processes is performed in [6] in the case $X = Y$ being a Brownian motion and in [10] for diffusion processes. In [5] path-dependent functionals are considered. For an overview the reader is referred to [4]. Actually our main result (Theorem 1) takes a more general form than (1):

- we consider an $L_q([0, T], \frac{dt}{T-t})$ -norm with $q \in [2, \infty]$ instead of the above L_∞ -norm with respect to $t \in [0, T]$;
- we consider an additional potential factor k in our parabolic problem to define v ;
- the change of measure, described in (1) by the change from X to Y , is described by Muckenhoupt weights;
- we also state results regarding the second derivatives.

Applications. The tight control of the behavior of the norms $\|\nabla v(t, X_t)\|_{L_2}$ as $t \rightarrow T$ is an issue that has been raised in [3], where the purpose was to analyze discrete approximations of stochastic integrals coming from the representation

$$g(X_T) = v(0, x_0) + \int_0^T \nabla v(t, X_t) \sigma(t, X_t) dB_t. \quad (2)$$

Discretizing the above stochastic integral and analyzing the resulting approximation error in L_2 , requires a better understanding how strongly the irregularity of the terminal function g transfers to the blow-up of the function $t \mapsto \|\nabla v(t, X_t)\|_{L_2}$ and higher derivatives of v as well. Major consequences of this analysis are the derivation of tight convergence rates for uniform time grids and the design of non-equidistant time grids to obtain optimal convergence rates.

Recently, similar results have been established in the context of Backward Stochastic Differential Equations [10, 5] to pave the way for the development of more efficient numerical schemes.

Finally, similar issues arise in the analysis of the Delta-Gamma hedging strategies in Finance, which typically result in a higher order approximation of the stochastic integral (2), see [11].

Within the applications in Stochastic Finance intrinsically two measures are involved: the historical measure for evaluating the risk, for example as L_p -mean, and the risk-neutral measure, under which the price and the hedging strategy are computed and which is related to the above function v . For this setting, the current results are particularly of interest. Moreover, the potential k may be interpreted as an interest rate.

2 Setting

Notation. We denote by $|\cdot|$ the Euclidean norm of a vector. Given a matrix C considered as operator $C : \ell_2^n \rightarrow \ell_2^N$, the expression $|C|$ stands for the Hilbert-Schmidt norm and C^\top for the transposed of C . The L_p -norm ($p \in [1, \infty]$) of a random vector $Z : \Omega \rightarrow \mathbb{R}^n$ or a random matrix $Z : \Omega \rightarrow \mathbb{R}^{n \times m}$ is denoted by $\|Z\|_p = \| |Z| \|_{L_p}$. As usual, $\partial_x^\alpha \varphi$ is the partial derivative of the order of a multi-index α (with length $|\alpha|$) with respect to $x \in \mathbb{R}^d$. The Hessian matrix of a function $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ is abbreviated by $D^2\varphi$ and the gradient (as row vector) by $\nabla\varphi$. In particular, this means that D^2 and ∇ always refer to the state variable $x \in \mathbb{R}^d$. If we mention that a constant depends on b , σ or k , then we implicitly indicate a possible dependence on T and d as well. Finally, letting $h : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{n \times m}$ we use the notation $\|h\|_\infty := \sup_{t,x} |h(t, x)|$.

The parabolic PDE. We fix $T > 0$ and consider the Cauchy problem

$$\begin{aligned} \mathcal{L}v &= 0 && \text{on } [0, T) \times \mathbb{R}^d, \\ v(T, x) &= g(x) \end{aligned}$$

with

$$\mathcal{L} := \partial_t + \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(t, x) \partial_{x_i, x_j}^2 + \sum_{i=1}^d b_i(t, x) \partial_{x_i} + k(t, x),$$

where $A := (a_{ij})_{ij} = \sigma \sigma^\top$. The assumptions on the coefficients and g are as follows:

- (C1) The functions $\sigma_{i,j}, b_i, k$ are bounded and belong to $C_b^{0,2}([0, T] \times \mathbb{R}^d)$ and there is some $\gamma \in (0, 1]$ such that the functions and their state-derivatives are γ -Hölder continuous with respect to the parabolic metric on each compactum of $[0, T] \times \mathbb{R}^d$. Moreover, σ is $1/2$ -Hölder continuous in t uniformly in x .
- (C2) $\sigma(t, x)$ is an invertible $d \times d$ -matrix with $\sup_{t,x} |\sigma^{-1}(t, x)| < +\infty$;
- (C3) the terminal function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is measurable and exponentially bounded: for some $K_g \geq 0$ and $\kappa_g \in [0, 2)$ we have

$$|g(x)| \leq K_g \exp(K_g |x|^{\kappa_g}) \quad \text{for all } x \in \mathbb{R}^d.$$

The condition (C2) implies that there exists a $\delta > 0$ with $\langle Ax, x \rangle \geq \delta |x|^2$ for all $x \in \mathbb{R}^d$, i.e. the operator \mathcal{L} is uniformly parabolic. Under the above assumptions there exists a fundamental solution:

Proposition 1 ([2, Theorem 7, p. 260; Theorem 10, pp. 72-74]). *Under the assumptions (C1) and (C2) there exists a fundamental solution $\Gamma(t, x; \tau, \xi) : \{0 \leq t < \tau \leq T\} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ for \mathcal{L} and a constant $c_{(3)} > 0$ such that for $0 \leq |a| + 2b \leq 3$ the derivatives $D_x^a D_t^b \Gamma$ exist in any order, are continuous, and satisfy*

$$|D_x^a D_t^b \Gamma(t, x; \tau, \xi)| \leq c_{(3)} (\tau - t)^{-\frac{|a|+2b}{2}} \gamma_{\tau-t}^d \left(\frac{x - \xi}{c_{(3)}} \right) \quad (3)$$

where $\gamma_s^d(x) := e^{-\frac{|x|^2}{2s}} / (\sqrt{2\pi s})^d$.

For

$$\begin{aligned} v(t, x) &:= \int_{\mathbb{R}^d} \Gamma(t, x; T, \xi) g(\xi) d\xi, \\ v(T, x) &:= g(x), \end{aligned}$$

and $0 \leq |a| + 2b \leq 3$ Proposition 1 implies that the derivatives $D_x^a D_t^b v$ exist in any order, are continuous on $[0, T) \times \mathbb{R}^d$ and satisfy

$$\begin{aligned} \mathcal{L}v &= 0 \quad \text{on } [0, T) \times \mathbb{R}^d, \\ |D_x^a D_t^b v(t, x)| &\leq c(T - t)^{-\frac{|a|+2b}{2}} \exp(c|x|^{\kappa_g}) \end{aligned}$$

for $x \in \mathbb{R}^d$ and $t \in [0, T)$, where $c > 0$ depends at most on $(\kappa_g, K_g, c_{(3)}, T)$.

The stochastic differential equation. Let $(B_t)_{t \in [0, T]}$ be a d -dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{Q})$, where $(\Omega, \mathcal{F}, \mathbb{Q})$ is complete, $(\mathcal{F}_t)_{t \in [0, T]}$ is right-continuous, $\mathcal{F} = \mathcal{F}_T$, \mathcal{F}_0 is generated by the null sets of \mathcal{F} and where all local martingales are continuous.

As we work on a closed time-interval we have to explain our understanding of a local martingale: we require that the localizing sequence of stopping times $0 \leq \tau_1 \leq \tau_2 \leq \dots \leq T$ satisfies $\lim_n \mathbb{Q}(\tau_n = T) = 1$. The reason for this is that we think about the extension of the filtration constantly by \mathcal{F}_T to (T, ∞) and that all local martingales $(N_t)_{t \in [0, T]}$ (in our setting) are extended by N_T to (T, ∞) . This yields the standard notion of a local martingale. However this is not needed explicitly in our paper, we only need this implicitly whenever we refer to results about the Muckenhoupt weights $A_\alpha(\mathbb{Q})$ from [13].

To shorten the notation, we denote sometimes the conditional expectation $\mathbb{E}(\cdot | \mathcal{F}_t)$ by $\mathbb{E}^{\mathcal{F}_t}(\cdot)$. The process $X = (X_t)_{t \in [0, T]}$ is given as strong unique solution of

$$X_t = x_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds.$$

Introducing the standing notation

$$K_t^X := e^{\int_0^t k(r, X_r) dr} \quad \text{and} \quad M_t := K_t^X v(t, X_t),$$

Itô's formula implies, for $t \in [0, T)$, that

$$M_t = v(0, x_0) + \int_0^t K_s^X \nabla v(s, X_s) \sigma(s, X_s) dB_s. \quad (4)$$

Moreover,

$$\lim_{t \rightarrow T} M_t = M_T \quad \text{and} \quad \lim_{t \rightarrow T} v(t, X_t) = g(X_T) \quad (5)$$

almost surely and in any $L_r(\mathbb{Q})$ with $r \in [1, \infty)$. Using Proposition 1 for $k = 0$ we also have

$$\mathbb{Q}(|X_t - x_0| > \lambda) \leq c \exp\left(-\frac{\lambda^2}{c}\right)$$

for all $\lambda \geq 0$ and $t \in [0, T]$, where $c > 0$ depends at most on (σ, b) and is, in particular, independent from the starting value $x_0 \in \mathbb{R}^d$. It directly implies that

$$g(X_T) \in \bigcap_{r \in [1, \infty)} L_r(\mathbb{Q})$$

so that Remark 1 applies as well. We will also use the following

Lemma 1 ([9], [10, Proof of Lemma 1.1], [5, Remark 3 in Appendix B]). *Let $t \in (0, T]$, $h : \mathbb{R}^d \rightarrow \mathbb{R}$ be a Borel function satisfying (C3) and Γ_X be the transition density of X , i.e. the function Γ from Proposition 1 in the case $k = 0$. Define*

$$H(s, x) := \int_{\mathbb{R}^d} \Gamma_X(s, x; t, \xi) h(\xi) d\xi \quad \text{for } (s, x) \in [0, t) \times \mathbb{R}^d.$$

For $r \in [0, t)$ and $x \in \mathbb{R}^d$ let $(Z_u)_{u \in [r, t]}$ be the diffusion based on (σ, b) starting in x defined on some $(M, \mathcal{G}, (\mathcal{G}_u)_{u \in [r, t]}, \mu)$ equipped with a standard $(\mathcal{G}_u)_{u \in [r, t]}$ -Brownian motion, where (M, \mathcal{G}, μ) is complete, $(\mathcal{G}_u)_{u \in [r, t]}$ is right-continuous and \mathcal{G}_r is generated by the null sets of \mathcal{G} . Then, for $q \in (1, \infty)$ and $s \in [r, t)$, one has a.s. that

$$\begin{aligned} |\nabla H(s, Z_s)| &\leq \kappa_q \frac{[\mathbb{E}(|h(Z_t) - \mathbb{E}(h(Z_t)|\mathcal{G}_s)|^q |\mathcal{G}_s)|]^{\frac{1}{q}}}{(t-s)^{\frac{1}{2}}}, \\ |D^2 H(s, Z_s)| &\leq \kappa_q \frac{[\mathbb{E}(|h(Z_t) - \mathbb{E}(h(Z_t)|\mathcal{G}_s)|^q |\mathcal{G}_s)|]^{\frac{1}{q}}}{t-s}, \end{aligned}$$

where $\kappa_q > 0$ depends at most on (σ, b, q) .

Conditions on the equivalent measure. In addition to the given measure \mathbb{Q} we will use an equivalent measure $\mathbb{P} \sim \mathbb{Q}$ and agree about the following standing assumption:

(P) There exists a martingale $Y = (Y_t)_{t \in [0, T]}$ with $Y_0 \equiv 0$ such that

$$\lambda_t := \mathcal{E}(Y)_t = e^{Y_t - \frac{1}{2}\langle Y \rangle_t} \quad \text{for } t \in [0, T]$$

is a martingale and

$$d\mathbb{P} = \lambda_T d\mathbb{Q}.$$

Definition 1. Assume that condition (P) is satisfied.

- (i) For $\alpha \in (1, \infty)$ we say that $\lambda_T \in A_\alpha(\mathbb{Q})$ provided that there is a constant $c > 0$ such that for all stopping times $\tau : \Omega \rightarrow [0, T]$ one has that

$$\mathbb{E}_{\mathbb{Q}} \left(\left| \frac{\lambda_\tau}{\lambda_T} \right|^{\frac{1}{\alpha-1}} \middle| \mathcal{F}_\tau \right) \leq c \quad \text{a.s.}$$

- (ii) For $\beta \in (1, \infty)$ we let $\lambda_T \in \mathcal{RH}_\beta(\mathbb{Q})$ provided that there is a constant $c > 0$ such that for all stopping times $\tau : \Omega \rightarrow [0, T]$ one has that

$$\mathbb{E}_{\mathbb{Q}}(|\lambda_T|^\beta | \mathcal{F}_\tau)^{\frac{1}{\beta}} \leq c \lambda_\tau \quad a.s.$$

The class $A_\alpha(\mathbb{Q})$ is the probabilistic variant of the Muckenhoupt condition and \mathcal{RH} stands for *reverse Hölder inequality*. Next we need

Definition 2. A martingale $Z = (Z_t)_{t \in [0, T]}$ is called BMO-martingale provided that $Z_0 \equiv 0$ and there is a $c > 0$ such that for all stopping times $\tau : \Omega \rightarrow [0, T]$ one has that

$$\mathbb{E}_{\mathbb{Q}}(|Z_T - Z_\tau|^2 | \mathcal{F}_\tau) \leq c^2 \quad a.s.$$

It is known [13, Theorems 2.3] that $(e^{Z_t - \frac{1}{2}\langle Z \rangle_t})_{t \in [0, T]}$ is a martingale provided that Z is a BMO-martingale.

Proposition 2 ([13, Theorems 2.4 and 3.4]). *Under condition (P) the following assertions are equivalent:*

- (i) Y is a BMO-martingale.
- (ii) $\mathcal{E}(Y) \in A_\alpha(\mathbb{Q})$ for some $\alpha \in (1, \infty)$.
- (iii) $\mathcal{E}(Y) \in \mathcal{RH}_\beta(\mathbb{Q})$ for some $\beta \in (1, \infty)$.

Remark 1. Under the assertions of Proposition 2 we have that $\lambda_T \in L_\beta(\mathbb{Q})$ and $1/\lambda_T \in L_{\alpha'}(\mathbb{P})$ with $1 = (1/\alpha) + (1/\alpha')$ so that

$$\bigcap_{r \in [1, \infty)} L_r(\mathbb{Q}) = \bigcap_{r \in [1, \infty)} L_r(\mathbb{P}).$$

Proposition 3 ([13, Theorems 2.3 and 3.19]). *Let Y be a BMO-martingale so that (P) is satisfied. For all $p \in (0, \infty)$ there is a $b_p(\mathbb{P}) > 0$ such that for all \mathbb{Q} -martingales N with $N_0 \equiv 0$ one has that*

$$\frac{1}{b_p(\mathbb{P})} \|N_T^*\|_{L_p(\mathbb{P})} \leq \|\sqrt{\langle N \rangle_T}\|_{L_p(\mathbb{P})} \leq b_p(\mathbb{P}) \|N_T^*\|_{L_p(\mathbb{P})}$$

where $N_t^* := \sup_{s \in [0, t]} |N_s|$.

An inequality. Given a probability space (M, Σ, μ) with a sub- σ algebra $\mathcal{G} \subseteq \Sigma$ and $Z \in L_p(M, \Sigma, \mu)$ with $p \in [1, \infty]$ we have that

$$\frac{1}{2} \|Z - \mathbb{E}(Z|\mathcal{G})\|_p \leq \inf_{Z' \in L_p(M, \mathcal{G}, \mu)} \|Z - Z'\|_p \leq \|Z - \mathbb{E}(Z|\mathcal{G})\|_p. \quad (6)$$

3 The result

In the following $\theta \in (0, 1]$ will be the main parameter of the fractional smoothness. Additionally, we introduce a fine-tuning parameter $q \in [2, \infty]$ and

$$\Phi_q(h) := \|h\|_{L_q([0, T], \frac{dt}{T-t})}$$

for a measurable function $h : [0, T) \rightarrow \mathbb{R}$. The aim of this paper is to prove the following result:

Theorem 1. *Let $p \in [2, \infty)$, $\alpha \in (1, p)$ and $\lambda_T \in A_\alpha(\mathbb{Q})$, and assume that (C1), (C2) and (P) are satisfied. Then, for $\theta \in (0, 1)$, $q \in [2, \infty]$, a measurable function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying (C3) and $d\mathbb{P} = \lambda_T d\mathbb{Q}$ the following assertions are equivalent:*

- (i) $_{\theta}$ $\Phi_q \left((T-t)^{-\frac{\theta}{2}} \|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)\|_{L_p(\mathbb{P})} \right) < +\infty.$
- (ii) $_{\theta}$ $\Phi_q \left((T-t)^{\frac{1-\theta}{2}} \|\nabla v(t, X_t)\|_{L_p(\mathbb{P})} \right) < +\infty.$
- (iii) $_{\theta}$ $\Phi_q \left((T-t)^{\frac{2-\theta}{2}} \|D^2 v(t, X_t)\|_{L_p(\mathbb{P})} \right) < +\infty.$

Remark 2. (1) Using [13, Corollary 3.3] it is sufficient to require that $\lambda_T \in A_p(\mathbb{Q})$ as in this case there is an $\varepsilon \in (0, p-1)$ such that $\lambda_T \in A_{p-\varepsilon}(\mathbb{Q})$. On the other hand, it would be of interest to investigate the case when $\lambda_T \in A_\alpha(\mathbb{Q})$ with $\alpha > p$. This is not done here.

- (2) Examples of functions g such that (i) $_{\theta}$ is satisfied are given for example in [3, 6, 7, 5].
- (3) In the case $X = B$, $\mathbb{P} = \mathbb{Q}$, $T = 1$ and $k = 0$ the conditions of Theorem 1 (neglecting the boundedness condition (C3)) are equivalent to that g belongs to the Malliavin Besov space $B_{p,q}^\theta$ on \mathbb{R}^d weighted by the standard Gaussian measure (see [8]).

- (4) The case $\theta = 1$ and $q \in [2, \infty)$ is not considered in Theorem 1 because it yields to pathologies: Let $X = B$, $\mathbb{P} = \mathbb{Q}$, $T = 1$ and $k = 0$. Condition (i_1) implies (ii_1) by Lemma 3 below. Moreover, condition (ii_1) and the monotonicity of $\|\nabla v(t, B_t)\|_{L_p(\mathbb{P})}$ ($(\nabla v(t, B_t))_{t \in [0,1]}$ is a martingale in this case) imply that $\nabla v(t, B_t) = 0$ a.s. so that $g(B_1)$ is almost surely constant (for example, one can use $g(B_1) = \mathbb{E}(g(B_1)) + \int_{[0,1]} \nabla v(t, B_t) dB_t$).
- (5) As the process $M = (M_t)_{t \in [0, T]}$ with $M_t = K_t^X v(t, X_t)$ is a martingale under \mathbb{Q} it is natural to consider condition (i_θ) for the corresponding martingale under \mathbb{P} as well:

$$(i'_\theta) \quad \Phi_q \left((T-t)^{-\frac{\theta}{2}} \|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})} \right) < +\infty.$$

One can easily check that $(i_\theta) \iff (i'_\theta)$ for $\theta \in (0, 1]$ and $q \in [1, \infty]$: Indeed, for any random variables U and V , respectively bounded and in $L_p(\mathbb{P})$, observe that

$$\begin{aligned} & \|UV - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(UV)\|_{L_p(\mathbb{P})} \\ & \leq \| [U - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} U] V \|_{L_p(\mathbb{P})} + \| \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(U) [V - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} V] \|_{L_p(\mathbb{P})} \\ & \quad + \| \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(U [\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(V) - V]) \|_{L_p(\mathbb{P})} \\ & \leq \| [U - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} U] V \|_{L_p(\mathbb{P})} + 2\|U\|_\infty \|V - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} V\|_{L_p(\mathbb{P})}. \end{aligned}$$

For $U := e^{\int_0^T k(r, X_r) dr}$ and $V := g(X_T)$ we have

$$|U - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} U| \leq 2\|k\|_\infty (T-t) e^{\|k\|_\infty T}$$

and obtain

$$\begin{aligned} & \| e^{\int_0^T k(r, X_r) dr} g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} (e^{\int_0^T k(r, X_r) dr} g(X_T)) \|_{L_p(\mathbb{P})} \\ & \leq 2e^{\|k\|_\infty T} \left[\|k\|_\infty (T-t) \|g(X_T)\|_{L_p(\mathbb{P})} + \|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)\|_{L_p(\mathbb{P})} \right]. \end{aligned}$$

This proves $(i_\theta) \implies (i'_\theta)$. The converse is proved similarly by letting $U := e^{-\int_0^T k(r, X_r) dr}$ and $V := e^{\int_0^T k(r, X_r) dr} g(X_T)$.

- (6) The case $\theta = 1$ and $q = \infty$.

(a) One has $(i'_1) \iff (ii_1) \implies (iii_1)$: First we observe that

$$\Phi_\infty \left((T-t)^{-\frac{1}{2}} \left(\int_t^T h(s)^2 ds \right)^{\frac{1}{2}} \right) \leq \Phi_\infty(h). \quad (7)$$

Then $(ii_1) \implies (i'_1)$ follows from (7) with $h(t) = \|\nabla v(t, X_t)\|_{L_p(\mathbb{P})}$ and Lemma 7. The implications $(i'_1) \implies (ii_1)$ and $(i_1) \implies (iii_1)$ follow by Lemmas 3 and 6.

(b) The implication $(iii_1) \implies (ii_1)$ is not true in general. Take $p = 2$, $q = \infty$, $X = B$, $\mathbb{P} = \mathbb{Q}$, $T = 1$, $k = 0$ and $d = 1$, then the counterexample $g(x) = \sqrt{x \vee 0}$ is discussed in [5].

(7) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a stochastic basis satisfying the usual conditions, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $(\mathcal{F}_t)_{t \in [0, T]}$ is right-continuous, \mathcal{F}_0 is generated by the null-sets of \mathcal{F} and where we can assume w.l.o.g. that $\mathcal{F} = \mathcal{F}_T$. Assume further that the filtration is obtained as augmentation of the natural filtration of a standard d -dimensional Brownian motion $W = (W_t)_{t \in [0, T]}$ starting in zero. It is known [15, Corollary 1 on p. 187] that on this stochastic basis all local martingales are continuous. Assume a progressively measurable d -dimensional process $\beta = (\beta_t)_{t \in [0, T]}$ with $\sup_{t, \omega} |\beta_t(\omega)| < \infty$ and consider the unique strong solution of the SDE

$$X_t = x_0 + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds - \int_0^t \beta_s ds.$$

Letting,

$$\begin{aligned} \gamma_s &:= \sigma^{-1}(s, X_s) \beta_s, \\ B_t &:= W_t - \int_0^t \gamma_s ds, \\ 1/\lambda_t &:= e^{\int_0^t \gamma_s^\top dW_s - \frac{1}{2} \int_0^t |\gamma_s|^2 ds} = e^{\int_0^t \gamma_s^\top dB_s + \frac{1}{2} \int_0^t |\gamma_s|^2 ds}, \\ d\mathbb{Q} &:= (1/\lambda_T) d\mathbb{P}, \end{aligned}$$

we obtain by the Girsanov Theorem that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{Q})$, $(B_t)_{t \in [0, T]}$ and $(X_t)_{t \in [0, T]}$ satisfy the assumptions of our paper (i.e. all martingales are continuous - which can be checked by expressing the conditional expectation under \mathbb{Q} by the conditional expectation under \mathbb{P} -, so that

local martingales are continuous as well) and that $\lambda_T \in A_\alpha$ for all $\alpha \in (1, \infty)$. Hence the passage from \mathbb{Q} to \mathbb{P} corresponds to adding a drift to the diffusion X .

- (8) In the case the drift term in item (7) is Markovian, i.e. $\beta_t = \beta(t, X_t)$ for an appropriate $\beta : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, and if we let $Y_t := v(t, X_t)$ and $Z_t := \nabla v(t, X_t)\sigma(t, X_t)$, then we get the BSDE

$$\begin{aligned} -dY_t &= [k(t, X_t)Y_t + Z_t\sigma^{-1}(t, X_t)\beta_t]dt - Z_t dW_t, \\ Y_T &= g(X_T). \end{aligned}$$

Then it is proved in [5] under certain conditions the equivalence between the following assertions for $p \in [2, \infty)$, $\theta \in (0, 1]$ and polynomially bounded g :

- (a) $\sup_{t \in [0, T]} (T - t)^{-\frac{\theta}{2}} \|g(X_T) - \mathbb{E}^{\mathcal{F}_t}(g(X_T))\|_{L_p(\mathbb{P})} < +\infty$.
- (b) $\sup_{t \in [0, T]} (T - t)^{\frac{1-\theta}{2}} \|Z_t\|_{L_p(\mathbb{P})} < +\infty$.

These are the analogues of (i_θ) and (ii_θ) for $q = \infty$.

4 Proof of Theorem 1

Through the whole section we assume that the condition (P) is satisfied.

4.1 Preliminaries

To estimate L_p norms under different measures, the following lemma is useful.

Lemma 2. *For any $1 < \alpha < p < \infty$, $\lambda_T \in A_\alpha(\mathbb{Q})$, $r := \frac{p}{p-\alpha}$, $U \in L_p(\Omega, \mathcal{F}, \mathbb{P})$, $V \in L_r(\Omega, \mathcal{F}, \mathbb{Q})$ and $c_{(8)} > 0$ such that $[\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}(|\frac{\lambda_t}{\lambda_T}|^{\frac{1}{\alpha-1}})]^{\frac{\alpha-1}{p}} \leq c_{(8)}$ a.s. we have that*

$$\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}|UV| \leq c_{(8)} [\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}|U|^p]^{\frac{1}{p}} [\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}|V|^r]^{\frac{1}{r}} \text{ a.s.} \quad (8)$$

Proof. Letting $1 = \frac{1}{p} + \frac{1}{p'} = \frac{1}{\alpha} + \frac{1}{\alpha'}$ one has a.s. that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}|UV| &= \lambda_t \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(|UV|/\lambda_T) \\ &\leq \lambda_t [\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}|U|^p]^{\frac{1}{p}} [\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(|V|^{p'} \lambda_T^{-\frac{p'}{r}} \lambda_T^{-p' + \frac{p'}{r}})]^{\frac{1}{p'}} \end{aligned}$$

$$\begin{aligned}
&\leq \lambda_t [\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} |U|^p]^{\frac{1}{p}} [\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} (|V|^r / \lambda_T)]^{\frac{1}{r}} [\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} \lambda_T^{-\alpha'}]^{\frac{r-p'}{p'r}} \\
&\leq c_{(8)} [\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} |U|^p]^{\frac{1}{p}} [\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |V|^r]^{\frac{1}{r}}.
\end{aligned}$$

□

As simple consequences of this lemma for $V \equiv 1$, observe that

$$\|\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} U\|_{L_p(\mathbb{P})} \leq c_{(8)} \|U\|_{L_p(\mathbb{P})} \quad \text{for } U \in L_p(\mathbb{P}). \quad (9)$$

In the next step we will estimate $\nabla v(t, X_t)$ and $D^2 v(t, X_t)$ in Lemmas 3 and 6 from above by conditional moments of $M_T = K_T^X g(X_T)$ and $g(X_T)$, and extend therefore Lemma 1 to the case $k = 0$ and allow a change of measure by Muckenhoupt weights.

Lemma 3. *For any $p \in (1, \infty)$, we have a.s. that*

$$|\nabla v(t, X_t)| \leq c_{(10)} \left[\frac{\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |M_T - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} M_T|^p \right)^{\frac{1}{p}}}{\sqrt{T-t}} + (T-t) \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |M_T|^p \right)^{\frac{1}{p}} \right], \quad (10)$$

where $c_{(10)} > 0$ depends at most on (σ, b, k, p) . The same estimate holds true if the measure \mathbb{Q} is replaced by the measure \mathbb{P} with $\lambda \in A_{\alpha}(\mathbb{Q})$ and $\alpha \in (1, p)$, where the constant $c_{(10)} > 0$ might additionally depend on \mathbb{Q} (and therefore implicitly on α).

Proof. The statement for \mathbb{P} for $p \in (1, \infty)$ can be deduced from the statement for \mathbb{Q} for $q \in (1, p)$. Let us fix $1 < q < p < \infty$, define $p_0 := p/q \in (1, \infty)$, take $r \in (p'_0, \infty)$ and let $\beta := \frac{p'_0 r - p'_0}{r - p'_0}$. For $\lambda \in A_{\alpha}(\mathbb{Q})$ with $1 = (1/\alpha) + (1/\beta)$ we apply Lemma 2 with p replaced by p_0 and get

$$(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |Z|^q)^{\frac{1}{q}} \leq c_{(8)}^{\frac{1}{q}} (\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} |Z|^p)^{\frac{1}{p}}$$

and, by (6),

$$(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |Z - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} Z|^q)^{\frac{1}{q}} \leq 2 (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |Z - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} Z|^q)^{\frac{1}{q}} \leq 2 c_{(8)}^{\frac{1}{q}} (\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} |Z - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} Z|^p)^{\frac{1}{p}}$$

whenever $Z \in \bigcap_{r \in [1, \infty)} L_r(\mathbb{Q})$ (cf. Remark 1). Because $\lim_{r \rightarrow \infty} \frac{p'_0 r - p'_0}{r - p'_0} = p'_0 = \frac{p}{p-q}$ and the convergence is from above, we can take β to be in $\left(\frac{p}{p-q}, \infty \right)$. Sending q to 1 gives that $\beta \in \left(\frac{p}{p-1}, \infty \right)$ or $\alpha \in (1, p)$.

Now we follow a martingale approach (see, for example, [9]) and prove the statement for the measure \mathbb{Q} .

(a) We define $(\nabla X_t)_{t \in [0, T]}$ to be the solution of a linear SDE (see [15, Chapter 5]):

$$\nabla X_t = I_d + \sum_{j=1}^d \int_0^t \nabla \sigma_j(s, X_s) \nabla X_s dB_s^j + \int_0^t \nabla b(s, X_s) \nabla X_s ds$$

and $\sigma(\cdot) = (\sigma_1(\cdot), \dots, \sigma_d(\cdot))$. This matrix-valued process is a.s. invertible and its inverse satisfies

$$\begin{aligned} [\nabla X_t]^{-1} &= I_d - \sum_{j=1}^d \int_0^t [\nabla X_s]^{-1} \nabla \sigma_j(s, X_s) dB_s^j \\ &\quad - \int_0^t [\nabla X_s]^{-1} (\nabla b(s, X_s) - \sum_{j=1}^d (\nabla \sigma_j(s, X_s))^2) ds. \end{aligned}$$

(b) Next we show that $(N_t)_{t \in [0, T]}$ with

$$N_t := K_t^X \nabla v(t, X_t) \nabla X_t + \left(\int_0^t \nabla k(s, X_s) \nabla X_s ds \right) M_t$$

is a \mathbb{Q} -martingale. One way consists in using Itô's formula to verify that N is a martingale. In fact, the bounded variation term in the Itô-process decomposition of N is

$$\int_0^t [K_s^X k(s, X_s) \nabla v(s, X_s) \nabla X_s + K_s^X C_s] ds + \int_0^t [\nabla k(s, X_s) \nabla X_s M_s] ds,$$

where $\int_0^t C_s ds$ is the bounded variation term of $\nabla v(t, X_t) \nabla X_t$. Hence it is sufficient to show that

$$C_s = -\nabla[v(s, X_s)k(s, X_s)] \nabla X_s.$$

The PDE for $w = \nabla v$ on $[0, T] \times \mathbb{R}^d$ reads as

$$\frac{\partial}{\partial t} w_i + \frac{1}{2} \langle A, D^2 w_i \rangle + \langle b, (\nabla w_i)^T \rangle = -\frac{1}{2} \langle \partial_{x_i} A, D^2 v \rangle - \langle \partial_{x_i} b, w^T \rangle - \partial_{x_i}(vk). \quad (11)$$

By a simple computation this gives that the bounded variation term of $(\sum_{i=1}^d \frac{\partial v}{\partial x_i}(t, X_t)(\nabla X_t)_{il})_{t \in [0, T]}$ computes as $-\sum_{i=1}^d \frac{\partial(vk)}{\partial x_i}(t, X_t)(\nabla X_s)_{il} dt$ and step (b) is complete.

(c) Exploiting the martingale property of N between t and some deterministic $S \in (t, T)$, we have

$$\begin{aligned}
& (S - t) \left[K_t^X \nabla v(t, X_t) \nabla X_t + \left(\int_0^t \nabla k(s, X_s) \nabla X_s ds \right) M_t \right] \\
&= \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \left(\int_t^S [K_r^X \nabla v(r, X_r) \nabla X_r + \left(\int_0^r \nabla k(s, X_s) \nabla X_s ds \right) M_r] dr \right) \\
&= \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \left(\left[\int_t^S K_r^X \nabla v(r, X_r) \sigma(r, X_r) dB_r \right] \left[\int_t^S (\sigma(r, X_r)^{-1} \nabla X_r)^\top dB_r \right]^\top \right) \\
&\quad + (S - t) M_t \left[\int_0^t \nabla k(s, X_s) \nabla X_s ds \right] \\
&\quad + \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \left(M_S \int_t^S \left[\int_t^r \nabla k(s, X_s) \nabla X_s ds \right] dr \right). \tag{12}
\end{aligned}$$

At the last equality, we have used the \mathbb{Q} -martingale property of $(M_t)_{t \in [0, T]}$ and the conditional Itô isometry

$$\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \left(\left[\int_t^S A_{1,r} dB_r \right] \left[\int_t^S A_{2,r} dB_r \right]^\top \right) = \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \left(\int_t^S A_{1,r} A_{2,r}^\top dr \right)$$

(available for any square integrable and progressively measurable matrix-valued processes $(A_{1,r})_r$ and $(A_{2,r})_r$, having d columns and an arbitrary number of rows). After simplifications, (12) writes

$$\begin{aligned}
& (S - t) K_t^X \nabla v(t, X_t) \nabla X_t \\
&= \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \left([M_S - M_t] \left[\int_t^S (\sigma(r, X_r)^{-1} \nabla X_r)^\top dB_r \right]^\top \right) \\
&\quad + \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \left(M_S \left[\int_t^S (S - s) \nabla k(s, X_s) \nabla X_s ds \right] \right).
\end{aligned}$$

Using that $M_S \rightarrow M_T$ in $L_2(\mathbb{Q})$ we derive

$$(T - t) K_t^X \nabla v(t, X_t)$$

$$\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \left([M_T - M_t] \left[\int_t^T (\sigma(r, X_r)^{-1} \nabla X_r [\nabla X_t]^{-1})^\top dB_r \right]^\top \right) \\
&\quad + \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \left(M_T \left[\int_t^T (T-s) \nabla k(s, X_s) \nabla X_s [\nabla X_t]^{-1} ds \right] \right).
\end{aligned}$$

Finally, observe that $\sup_{t \in [0, T]} \sup_{r \in [t, T]} \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} (|\nabla X_r [\nabla X_t]^{-1}|^q)$ is a bounded random variable for any $q \geq 1$; therefore, standard computations complete our assertion. \square

For the following we let $m(t, x) := v(t, x)k(t, x)$.

Lemma 4. *For $0 \leq r < t \leq T$ and $1 < p_0 < p < \infty$ one has a.s. that*

$$\begin{aligned}
&(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |m(t, X_t) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} m(t, X_t)|^{p_0})^{\frac{1}{p_0}} \\
&\leq c_{(13)} \left[\sqrt{t-r} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M^*|^p)^{\frac{1}{p}} + (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M_t - M_r|^{p_0})^{\frac{1}{p_0}} \right] \quad (13)
\end{aligned}$$

where $M^* := \sup_{s \in [0, T]} |M_s|$ and $c_{(13)} > 0$ depends at most on (p_0, p, σ, b, k) .

Proof. (a) For $\frac{1}{p_0} = \frac{1}{q_k} + \frac{1}{r_k} = \frac{1}{s_k} + \frac{1}{t_k} + \frac{1}{r_k}$ with $r_k, s_k, t_k \in [p_0, \infty]$, a sub- σ -algebra $\mathcal{G} \subseteq \mathcal{F}$, $\underline{U}_k := U_1 \cdots U_k$ and $\overline{U}_k := U_k \cdots U_N$ with $\underline{U}_0 := 1$ and $\overline{U}_{N+1} := 1$, and for $\underline{U}_{k-1} \in L_{t_k}(\mathbb{Q})$, $U_k \in L_{s_k}(\mathbb{Q})$, $\overline{U}_{k+1} \in L_{r_k}(\mathbb{Q})$, where $k = 1, \dots, N$, we get by a telescoping sum argument and the conditional Hölder inequality that

$$\begin{aligned}
&(\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} |U_1 \cdots U_N - E_{\mathbb{Q}}^{\mathcal{G}}(U_1 \cdots U_N)|^{p_0})^{\frac{1}{p_0}} \\
&\leq \sum_{k=1}^N (\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} |[\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}}(\underline{U}_{k-1})] U_k - \mathbb{E}_{\mathbb{Q}}^{\mathcal{G}}(\underline{U}_k)|^{q_k})^{\frac{1}{q_k}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} |\overline{U}_{k+1}|^{r_k})^{\frac{1}{r_k}} \\
&\leq \sum_{k=1}^N (\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} |[\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}}(\underline{U}_{k-1})] U_k - \mathbb{E}_{\mathbb{Q}}^{\mathcal{G}}(\underline{U}_{k-1}) \mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} U_k|^{q_k})^{\frac{1}{q_k}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} |\overline{U}_{k+1}|^{r_k})^{\frac{1}{r_k}} \\
&\quad + \sum_{k=1}^N (\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} |[\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}}(\underline{U}_{k-1})] \mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} U_k - [\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}}(\underline{U}_k)]|^{q_k})^{\frac{1}{q_k}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} |\overline{U}_{k+1}|^{r_k})^{\frac{1}{r_k}} \\
&\leq 2 \sum_{k=1}^N (\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} |\underline{U}_{k-1}|^{t_k})^{\frac{1}{t_k}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} |U_k - \mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} U_k|^{s_k})^{\frac{1}{s_k}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{G}} |\overline{U}_{k+1}|^{r_k})^{\frac{1}{r_k}}.
\end{aligned}$$

(b) We apply (a) to $N = 3$ and $m(s, X_s) = k(s, X_s)(K_s^X)^{-1}M_s$ to derive

$$\begin{aligned}
& (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |m(t, X_t) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} m(t, X_t)|^{p_0})^{\frac{1}{p_0}} \\
& \leq 2\|k\|_{\infty} e^{T\|k\|_{\infty}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M_t - M_r|^{p_0})^{\frac{1}{p_0}} \\
& \quad + 2 (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |k(t, X_t) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} k(t, X_t)|^{\beta})^{\frac{1}{\beta}} e^{T\|k\|_{\infty}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M^*|^p)^{\frac{1}{p}} \\
& \quad + 2\|k\|_{\infty} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |(K_t^X)^{-1} - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} (K_t^X)^{-1}|^{\beta})^{\frac{1}{\beta}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M^*|^p)^{\frac{1}{p}}
\end{aligned}$$

for $\frac{1}{p_0} = \frac{1}{p} + \frac{1}{\beta}$. We conclude by

$$\begin{aligned}
(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |k(t, X_t) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} k(t, X_t)|^{\beta})^{\frac{1}{\beta}} & \leq 2 (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |k(t, X_t) - k(t, X_r)|^{\beta})^{\frac{1}{\beta}} \\
& \leq 2\|\nabla k\|_{\infty} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |X_t - X_r|^{\beta})^{\frac{1}{\beta}} \\
& \leq 2\|\nabla k\|_{\infty} c(b, \sigma, \beta) \sqrt{t - r}
\end{aligned}$$

and $(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |(K_t^X)^{-1} - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} (K_t^X)^{-1}|^{\beta})^{\frac{1}{\beta}} \leq 2\|k\|_{\infty} (t - r) e^{T\|k\|_{\infty}}$. \square

Lemma 5. For $0 \leq r < t < T$ and $p \in (1, \infty)$ one has a.s. that

$$(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M_t - M_r|^p)^{\frac{1}{p}} \leq c_{(14)} \left[\left(\frac{t - r}{T - t} \right)^{\frac{1}{2}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M_T - M_r|^p)^{\frac{1}{p}} + (t - r)^{\frac{1}{2}} |M_r| \right] \quad (14)$$

where $c_{(14)} \geq 1$ depends at most on (p, σ, b, k) .

Proof. Let $p_0 := \frac{1+p}{2}$, $\lambda_u := K_u^X \nabla v(u, X_u) \sigma(u, X_u)$ and $0 \leq r \leq u \leq t$. Then Lemma 3 implies that

$$\begin{aligned}
& |\lambda_u| e^{-T\|k\|_{\infty}} \\
& \leq \|\sigma\|_{\infty C_{(10), p_0}} \left[(T - u)^{-\frac{1}{2}} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u} |M_T - M_u|^{p_0} \right)^{\frac{1}{p_0}} \right. \\
& \quad \left. + (T - u) \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u} |M_T|^{p_0} \right)^{\frac{1}{p_0}} \right] \\
& \leq \|\sigma\|_{\infty C_{(10), p_0}} \left[(T - u)^{-\frac{1}{2}} 2 \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u} |M_T - M_r|^{p_0} \right)^{\frac{1}{p_0}} \right. \\
& \quad \left. + (T - u) \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u} |M_T - M_r|^{p_0} \right)^{\frac{1}{p_0}} + (T - u) |M_r| \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \|\sigma\|_{\infty \mathcal{C}(10), p_0} \left[[2 + T^{\frac{3}{2}}] (T - u)^{-\frac{1}{2}} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u} |M_T - M_r|^{p_0} \right)^{\frac{1}{p_0}} + (T - u) |M_r| \right] \\
&\leq \|\sigma\|_{\infty \mathcal{C}(10), p_0} [2 + T^{\frac{3}{2}} + T] \left[(T - t)^{-\frac{1}{2}} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u} |M_T - M_r|^{p_0} \right)^{\frac{1}{p_0}} + |M_r| \right].
\end{aligned}$$

Letting $c := e^{T\|k\|_{\infty}} \|\sigma\|_{\infty \mathcal{C}(10), p_0} [2 + T^{\frac{3}{2}} + T]$ we conclude the proof by using the Burkholder-Davis-Gundy inequalities in order to get

$$\begin{aligned}
&\frac{1}{a_p} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M_t - M_r|^p \right)^{\frac{1}{p}} \\
&\leq \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} \left(\int_r^t |\lambda_u|^2 du \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\
&\leq c \left[(T - t)^{-\frac{1}{2}} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} \left(\int_r^t \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u} |M_T - M_r|^{p_0} \right)^{\frac{2}{p_0}} du \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right. \\
&\quad \left. + \sqrt{t - r} |M_r| \right] \\
&\leq c \left[\sqrt{\frac{t - r}{T - t}} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} \left(\sup_{u \in [r, t]} \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u} |M_T - M_r|^{p_0} \right)^{\frac{p}{p_0}} \right)^{\frac{1}{p}} + \sqrt{t - r} |M_r| \right] \\
&\leq c \left[\left(\frac{p/p_0}{(p/p_0) - 1} \right)^{\frac{1}{p_0}} \sqrt{\frac{t - r}{T - t}} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |M_T - M_r|^{p_0} \right)^{\frac{p}{p_0}} \right)^{\frac{1}{p}} \right. \\
&\quad \left. + \sqrt{t - r} |M_r| \right] \\
&\leq c \left[\left(\frac{p}{p - p_0} \right)^{\frac{1}{p_0}} \sqrt{\frac{t - r}{T - t}} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |M_T - M_r|^p \right)^{\frac{1}{p}} + \sqrt{t - r} |M_r| \right].
\end{aligned}$$

□

Lemma 6. For $p \in (1, \infty)$ there is a constant $c_{(15)} = c(\sigma, b, k, p) > 0$ such that one has a.s. that

$$|D^2 v(r, X_r)| \leq c_{(15)} \left[\frac{\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |g(X_T) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} g(X_T)|^p \right)^{\frac{1}{p}}}{T - r} + \sqrt{T - r} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M^*|^p \right)^{\frac{1}{p}} \right]. \quad (15)$$

The same estimate holds true if the measure \mathbb{Q} is replaced by the measure \mathbb{P} with $\lambda \in A_\alpha(\mathbb{Q})$ and $\alpha \in (1, p)$, where the constant $c_{(15)} > 0$ might additionally depend on \mathbb{Q} (and therefore implicitly on α).

Proof. (a) The statement for \mathbb{P} for $p \in (1, \infty)$ can be deduced from the statement for \mathbb{Q} for $q \in (1, p)$ as in the first step of the proof of Lemma 3.

(b) Now we show the estimate for the measure \mathbb{Q} . For $0 \leq s \leq t \leq T$, a fixed $T_0 \in (0, T)$ and $r \in [0, T_0]$ we let

$$v^t(s, x) := \mathbb{E}_{\mathbb{Q}}(m(t, X_t) | X_s = x) \quad \text{and} \quad v_h(r, x) := \mathbb{E}_{\mathbb{Q}}(v(T_0, X_{T_0}) | X_r = x).$$

Itô's formula applied to v gives for $r \in [0, T_0]$ that

$$v(r, x) = \mathbb{E}_{\mathbb{Q}} \left(v(T_0, X_{T_0}) + \int_r^{T_0} (kv)(t, X_t) dt | X_r = x \right)$$

and therefore

$$v(r, x) = v_h(r, x) + \int_r^{T_0} v^t(r, x) dt.$$

Using Lemma 1 and the arguments from Remark 2(5) one can show for $0 \leq r < t \leq T_0 < T$ that

$$|\nabla v^t(r, x)| \leq \gamma e^{\gamma|x|^{k_g}} \quad \text{and} \quad |D^2 v^t(r, x)| \leq \frac{\gamma}{\sqrt{t-r}} e^{\gamma|x|^{k_g}}, \quad (16)$$

where $\gamma > 0$ depends at most on $(\sigma, b, k, K_g, k_g, T_0)$. From this we deduce that

$$D^2 v(r, x) = D^2 v_h(r, x) + \int_r^{T_0} D^2 v^t(r, x) dt$$

where (16) are used to interchange the integral and D^2 . For $p_0 := \frac{1+p}{2}$, $0 \leq r < t \leq T$ and $s \in [0, T_0]$ we again use Lemma 1 to get

$$\begin{aligned} |D^2 v^t(r, X_r)| &\leq \frac{\kappa_{p_0}}{(t-r)} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |m(t, X_t) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} m(t, X_t)|^{p_0} \right)^{\frac{1}{p_0}} \quad \text{a.s.}, \\ |D^2 v_h(s, X_s)| &\leq \frac{\kappa_p}{(T_0-s)} \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_s} |v(T_0, X_{T_0}) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_s} v(T_0, X_{T_0})|^p \right)^{\frac{1}{p}} \quad \text{a.s.} \end{aligned}$$

From the first estimate we derive by Lemmas 4 and 5 (with p replaced by p_0) a.s. that

$$|D^2 v^t(r, X_r)|$$

$$\begin{aligned}
&\leq \frac{\kappa_{p_0}}{(t-r)} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |m(t, X_t) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} m(t, X_t)|^{p_0})^{\frac{1}{p_0}} \\
&\leq \frac{\kappa_{p_0} c_{(13)}}{(t-r)} \left[\sqrt{t-r} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M^*|^p)^{\frac{1}{p}} + (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M_t - M_r|^{p_0})^{\frac{1}{p_0}} \right] \\
&\leq \kappa_{p_0} c_{(13)} [1 + c_{(14)}] \frac{1}{\sqrt{t-r}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M^*|^p)^{\frac{1}{p}} \\
&\quad + \kappa_{p_0} c_{(13)} c_{(14)} \frac{1}{\sqrt{T-t}\sqrt{t-r}} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M_T - M_r|^{p_0})^{\frac{1}{p_0}}
\end{aligned}$$

and

$$\int_r^T |D^2 v^t(r, X_r)| dt \leq c \left[\sqrt{T-r} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M^*|^p)^{\frac{1}{p}} + (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M_T - M_r|^p)^{\frac{1}{p}} \right]$$

with $c := \kappa_{p_0} c_{(13)} \max\{2 + 2c_{(14)}, c_{(14)} \text{Beta}(\frac{1}{2}, \frac{1}{2})\}$. The second estimate yields by $T_0 \uparrow T$ and (5) that

$$|D^2 v_h(r, X_r)| \leq \frac{\kappa_p}{(T-r)} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |g(X_T) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} g(X_T)|^p)^{\frac{1}{p}}$$

and the upper bound is independent of T_0 . Combining the estimates with

$$\begin{aligned}
&(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M_T - M_r|^p)^{\frac{1}{p}} \leq 2e^{\|k\|_{\infty} T} \\
&\left[\|k\|_{\infty} (T-r) e^{\|k\|_{\infty} T} (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |M^*|^p)^{\frac{1}{p}} + (\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} |g(X_T) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} g(X_T)|^p)^{\frac{1}{p}} \right]
\end{aligned}$$

using the arguments from Remark 2(5) the proof is complete. \square

Lemma 7. *Let $\lambda = \mathcal{E}(Y)$, where Y is a BMO-martingale with $Y_0 = 0$. Then, for $p \in (1, \infty)$, $t \in [0, T]$ and $c_{(17)} := 2b_p(\mathbb{P})e^{T\|k\|_{\infty}}\|\sigma\|_{\infty}$ we have that*

$$\|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})} \leq c_{(17)} \left\| \left(\int_t^T |\nabla v(s, X_s)|^2 ds \right)^{\frac{1}{2}} \right\|_{L_p(\mathbb{P})}. \quad (17)$$

Proof. Owing to inequality (6) and applying Proposition 3, we get

$$\begin{aligned}
\|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})} &\leq 2\|M_T - M_t\|_{L_p(\mathbb{P})} \\
&\leq 2b_p(\mathbb{P}) \left\| \sqrt{\langle M \rangle_T - \langle M \rangle_t} \right\|_{L_p(\mathbb{P})}
\end{aligned}$$

$$= 2b_p(\mathbb{P}) \left\| \sqrt{\int_t^T |K_s^X \nabla v(s, X_s) \sigma(s, X_s)|^2 ds} \right\|_{L_p(\mathbb{P})}.$$

□

Lemma 8. For $p \in [2, \infty)$, $\lambda_T \in A_\alpha(\mathbb{Q})$ with $\alpha \in (1, p)$, $0 \leq s < t < T$ and $l = 1, \dots, d$ we have that

$$\begin{aligned} & \|K_t^X \partial_{x_l} v(t, X_t) - K_s^X \partial_{x_l} v(s, X_s)\|_{L_p(\mathbb{P})} \\ & \leq c_{(18)} \left[\|M_T\|_{L_p(\mathbb{P})} \int_s^t \frac{dr}{\sqrt{T-r}} + \left(\int_s^t \|D^2 v(r, X_r)\|_{L_p(\mathbb{P})}^2 dr \right)^{\frac{1}{2}} \right] \end{aligned} \quad (18)$$

with $c_{(18)} > 0$ depending at most on $(\sigma, b, k, p, \mathbb{P})$ (and therefore implicitly on α).

Proof. Exploiting (11) and Propositions 2 and 3 we get that

$$\begin{aligned} & \|K_t^X \partial_{x_l} v(t, X_t) - K_s^X \partial_{x_l} v(s, X_s)\|_{L_p(\mathbb{P})} \\ & \leq \left\| \int_s^t K_r^X (\nabla \partial_{x_l} v)(r, X_r) \sigma(r, X_r) dB_r \right\|_{L_p(\mathbb{P})} \\ & \quad + \left\| \int_s^t K_r^X \left[\frac{1}{2} |\langle \partial_{x_l} A(r, X_r), D^2(r, X_r) \rangle| + |\langle \partial_{x_l} b(r, X_r), \nabla v(r, X_r)^\top \rangle| \right. \right. \\ & \quad \left. \left. + |(\partial_{x_l} k)(r, X_r) v(r, X_r)| \right] dr \right\|_{L_p(\mathbb{P})} \\ & \leq b_p(\mathbb{P}) \left\| \left(\int_s^t |K_r^X (\nabla \partial_{x_l} v)(r, X_r) \sigma(r, X_r)|^2 dr \right)^{\frac{1}{2}} \right\|_{L_p(\mathbb{P})} \\ & \quad + \frac{1}{2} \|\partial_{x_l} A\|_\infty \left\| \int_s^t |K_r^X D^2 v(r, X_r)| dr \right\|_{L_p(\mathbb{P})} \\ & \quad + \|\partial_{x_l} b\|_\infty \left\| \int_s^t |K_r^X \nabla v(r, X_r)| dr \right\|_{L_p(\mathbb{P})} \\ & \quad + \|\partial_{x_l} k\|_\infty \left\| \int_s^t |K_r^X v(r, X_r)| dr \right\|_{L_p(\mathbb{P})}. \end{aligned}$$

Inequality (9) directly yields

$$\sup_{r \in [0, T]} \|K_r^X v(r, X_r)\|_{L_p(\mathbb{P})} = \sup_{r \in [0, T]} \|\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} M_T\|_{L_p(\mathbb{P})} \leq c_{(8)} \|M_T\|_{L_p(\mathbb{P})}.$$

Moreover, by Lemma 3,

$$\|\nabla v(r, X_r)\|_{L_p(\mathbb{P})} \leq c_{(10)} (T - r)^{-\frac{1}{2}} (2 + T^{3/2}) \|M_T\|_{L_p(\mathbb{P})}.$$

Inserting these estimates in the above upper bound for

$$\|K_t^X \partial_{x_l} v(t, X_t) - K_s^X \partial_{x_l} v(s, X_s)\|_{L_p(\mathbb{P})}$$

gives the announced result. \square

Lemma 9 ([8, Proposition A.4]). *Let $0 < \theta < 1$, $2 \leq q \leq \infty$ and $d^k : [0, T) \rightarrow [0, \infty)$, $k = 0, 1, 2$, be measurable functions. Assume that there are $A \geq 0$ and $D \geq 1$ such that*

$$\begin{aligned} \frac{1}{D} (T - t)^{\frac{k}{2}} d^k(t) &\leq d^0(t) \leq D \left(\int_t^T [d^1(s)]^2 ds \right)^{\frac{1}{2}}, \\ d^1(t) &\leq A + D \left(\int_0^t [d^2(u)]^2 du \right)^{\frac{1}{2}} \end{aligned}$$

for $k = 1, 2$ and $t \in [0, T)$. Then there is a constant $c_{(19)} > 0$, depending at most on (D, θ, q, T) , such that, for $k, l \in \{0, 1, 2\}$,

$$A + \Phi_q \left((T - t)^{\frac{k-\theta}{2}} d^k(t) \right) \sim_{c_{(19)}} A + \Phi_q \left((T - t)^{\frac{l-\theta}{2}} d^l(t) \right). \quad (19)$$

4.2 Proof of Theorem 1

We let

$$\begin{aligned} d^0(t) &:= \sqrt{T - t} + \|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})}, \\ d^1(t) &:= 1 + \|\nabla v(t, X_t)\|_{L_p(\mathbb{P})}, \\ d^2(t) &:= 1 + \|D^2 v(t, X_t)\|_{L_p(\mathbb{P})}. \end{aligned}$$

From Lemma 3 we get that

$$d^1(t) = 1 + \|\nabla v(t, X_t)\|_{L_p(\mathbb{P})}$$

$$\begin{aligned}
&\leq 1 + c_{(10)}(T-t)^{-\frac{1}{2}}\|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})} + c_{(10)}(T-t)\|M_T\|_{L_p(\mathbb{P})} \\
&\leq (T-t)^{-\frac{1}{2}}[1 + c_{(10)} + c_{(10)}T\|M_T\|_{L_p(\mathbb{P})}] \\
&\quad \left[\sqrt{T-t} + \|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})} \right] \\
&= (T-t)^{-\frac{1}{2}}[1 + c_{(10)} + c_{(10)}T\|M_T\|_{L_p(\mathbb{P})}]d^0(t).
\end{aligned}$$

From Lemma 6 we get that

$$\begin{aligned}
d^2(t) &= 1 + \|D^2v(t, X_t)\|_{L_p(\mathbb{P})} \\
&\leq 1 + c_{(15)} \left[\frac{\|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)\|_{L_p(\mathbb{P})}}{T-t} + \sqrt{T-t}\|M^*\|_{L_p(\mathbb{P})} \right].
\end{aligned}$$

Using Remark 2(5) we have that

$$\begin{aligned}
&\|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)\|_{L_p(\mathbb{P})} \\
&\leq 2e^{\|k\|_{\infty}T} [\|k\|_{\infty}(T-t)\|M_T\|_{L_p(\mathbb{P})} + \|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})}].
\end{aligned}$$

Together with the previous estimate we obtain a constant $c > 0$ depending at most on $(c_{(15)}, k, T, \|M^*\|_{L_p(\mathbb{P})})$ such that

$$d^2(t) \leq c(T-t)^{-1}d^0(t).$$

From Lemma 7 we get that

$$\begin{aligned}
d^0(t) &= \sqrt{T-t} + \|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})} \\
&\leq \sqrt{T-t} + c_{(17)} \left(\int_t^T \|\nabla v(s, X_s)\|_{L_p(\mathbb{P})}^2 ds \right)^{\frac{1}{2}} \\
&\leq [1 + c_{(17)}] \left(\int_t^T [1 + \|\nabla v(s, X_s)\|_{L_p(\mathbb{P})}]^2 ds \right)^{\frac{1}{2}} \\
&= [1 + c_{(17)}] \left(\int_t^T [d^1(s)]^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Finally, from Lemma 8 for $s = 0$ we deduce that

$$\begin{aligned}
d^1(t) &= 1 + \|\nabla v(t, X_t)\|_{L_p(\mathbb{P})} \\
&\leq 1 + e^{\|k\|_{\infty}T} \|K_t^X \nabla v(t, X_t)\|_{L_p(\mathbb{P})} \\
&\leq 1 + e^{\|k\|_{\infty}T} \|K_0^X \nabla v(0, X_0)\|_{L_p(\mathbb{P})}
\end{aligned}$$

$$\begin{aligned}
& + e^{\|k\|_\infty T} c_{(18)} \sqrt{d} \left[\|M_T\|_{L_p(\mathbb{P})} 2\sqrt{T} + \left(\int_0^t \|D^2 v(r, X_r)\|_{L_p(\mathbb{P})}^2 dr \right)^{\frac{1}{2}} \right] \\
& \leq d_1 + d_2 \left(\int_0^t \|D^2 v(r, X_r)\|_{L_p(\mathbb{P})}^2 dr \right)^{\frac{1}{2}} \\
& \leq d_1 + d_2 \left(\int_0^t [d^2(r)]^2 dr \right)^{\frac{1}{2}}
\end{aligned}$$

with

$$\begin{aligned}
d_1 &:= 1 + e^{\|k\|_\infty T} \left[\|K_0^X \nabla v(0, X_0)\|_{L_p(\mathbb{P})} + 2c_{(18)} \sqrt{dT} \|M_T\|_{L_p(\mathbb{P})} \right], \\
d_2 &:= e^{\|k\|_\infty T} c_{(18)} \sqrt{d}.
\end{aligned}$$

Lemma 9 combined with Remark 2(5) yields the statement. \square

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